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LETTER TO THE EDITOR

A note on quantum braking

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Abstract. In a superfluid at $T = 0°$ K, the zero-point quantum fluctuations give a non-vanishing amplitude to the phonons. The scattering of these phonons by a moving obstacle at low speed yields a drag that is proportional to the speed. This drag is computed for a sphere in some detail and the final result shows a very strong divergence, as predicted recently by Pomeau.

This letter concerns a detailed calculation of the (formally diverging) drag by zero-point phonons on a moving obstacle in a quantum fluid. The physical ideas behind this have been explained in [1], and we present here a calculation of this force (see below for the meaning). Although this calculation in itself is without mystery, some conclusions of [1] have been challenged on the grounds that they go against the accepted views on irreversibility in superfluids (something that was shown in [1] not to hold however), and the possibility has been raised that, since [1] is about order-of-magnitude estimates, it could be that the numerical coefficients of the laws so derived are zero. Thus, it is of interest to look more precisely at this matter, including a study of the numerical coefficients.

The starting point is the Gross–Pitaevskii (GP) [2] equation for the condensate wavefunction Ψ:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + g|\Psi|^2\Psi \tag{1}$$

where m is the mass of the particles, $\hbar/2\pi$ the Planck constant and g is a (positive) interaction coefficient. The classical field Ψ will be quantized later on. Transport equations follow from (1) for the number density $\rho = |\Psi|^2$ and the mass current $\boldsymbol{J}$ (bold type is used for vectors) whose Cartesian component J_μ are

$$J_\mu = \frac{i\hbar}{2}[\Psi\partial_\mu\Psi^* - \Psi^*\partial_\mu\Psi]$$

with $\partial_\mu = \frac{\partial}{\partial x_\mu}$ and Ψ^* complex conjugate of Ψ. The momentum conservation reads

$$\frac{\partial J_\mu}{\partial t} + \partial_\nu T_{\mu\nu} = 0$$

where $T_{\mu\nu}$ is the tensor of flux of the momentum

$$-\frac{\hbar^2}{4m}[\Psi\partial_{\mu\nu}\Psi^* + \Psi^*\partial_{\mu\nu}\Psi - \partial_\mu\Psi^*\partial_\nu\Psi - \partial_\mu\Psi\partial_\nu\Psi^*] + \frac{g\hbar}{2}\delta_{\nu\mu}|\Psi|^4$$

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where $\delta_{\nu\mu}$ is the Kronecker delta. The force F_μ on a body is the integral of $T_{\mu\nu}$ on its surface:

$$F_\mu = \int \mathrm{d}\Sigma \, n_\nu T_{\mu\nu}$$

where n_ν is the unit normal to the surface, $\mathrm{d}\Sigma$ the area element, and the Einstein convention is used for index summation. From this, one may compute the force on a surface due to the reflection of a sound wave (a kind of radiation pressure). Let Ψ_0 be the wavefunction of the uniform ground state. Linear perturbations to the ground state are propagating waves ('phonons') such that each Fourier component behaves like

$$\delta\Psi_k = \Psi_0 \eta \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{e}^{\mathrm{i}(\omega+\Omega^0)t}$$

where η is a small dimensionless number and where Ω^0 is the ground-state frequency $-(g|\Psi_0|^2)/\hbar$ (that we shall drop later on, because it is unobservable). The frequency ω is related [3] to the wavenumber k by

$$\omega^2 = \frac{\hbar^2 k^4}{4m^2} + c_s^2 k^2$$

where $c_s^2 = g\rho_0/m$ is the square of the speed of sound and $\rho_0 = |\Psi_0|^2$ the ground-state mass density.

The calculation done below is carried out in two steps. First we shall consider the reflection of a plane wave by a plane surface. This is relevant for the problem under consideration because we shall assume that the wavelength (as shown in [1]) is much shorter than the radius of curvature of the obstacle. This plane surface can be replaced in a first approximation by the local tangent plane. Then we calculate the pressure of the wave on the reflecting surface and at last obtain the total force by integrating over the surface of the moving obstacle and by taking the amplitude of the phonons as given by the zero-point fluctuations. The final result (equation (8)) is given by an integral with a bulky 'ultraviolet divergence'. In [1] it was explained how to deal with this divergence; this topic will be considered in more detail in a future paper.

Let a plane sound wave come from $z < 0$ and be reflected by the plane $z = 0$ with the boundary condition $\delta\Psi = 0$ at $z = 0$. The solution of the linearized GP equation is the sum of an incident wave $\delta\Psi_\mathrm{i} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{e}^{-\mathrm{i}\omega t}$ and of a reflected wave $\delta\Psi_\mathrm{r} \mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}} \mathrm{e}^{-\mathrm{i}\omega t}$. The Cartesian components of the wavevector of the incident ($\boldsymbol{k}$) and reflected ($\boldsymbol{k}'$) wave can be written as

$$\boldsymbol{k} = (k_x, 0, q) \qquad \boldsymbol{k}' = (k_x, 0, q' = -q) \qquad \omega = c_s (k_x^2 + q^2)^{1/2}.$$

Notice that these Descartes-like reflection formulae assume a homogeneous medium which is not absolutely true, due to a boundary layer near $z = 0$ of microscopic thickness $\lambda_0 = h/(mc_s)$; this is neglected because it is smaller than the wavelength of the phonons under consideration.

Thus, within the various assumptions made, only one contribution quadratic in the amplitude of the sound wave to $T_{\mu\nu}$ does not vanish on the $z = 0$ boundary:

$$T_{zz} = \frac{2\hbar^2 q^2}{m} |\delta\Psi_\mathrm{i}|^2.$$

With intrinsic notation this can be written as

$$T_{nn} = \frac{2\hbar^2}{m} |\delta\Psi_\mathrm{i}|^2 (\boldsymbol{k}\cdot\boldsymbol{n})^2 \tag{2}$$

where $\boldsymbol{n}$ is the unit normal to the reflecting surface. The force $\boldsymbol{F}$ arising from the radiation pressure of a single impinging plane-wave is obtained by integrating the flux T_{nn} across the surface of the obstacle. This is given by the surface integral

$$\boldsymbol{F} = \int \mathrm{d}\Sigma\, \boldsymbol{n} T_{nn} H(\boldsymbol{k}\cdot\boldsymbol{n}) \tag{3}$$

where $H(\boldsymbol{k}\cdot\boldsymbol{n})$ is the Heaviside function imposing the condition that the wave is incoming. A non-convex surface allowing multiscattering would complicate these formulae; we shall not consider this possibility.

We have not yet considered the perturbation to the phonon field due to the motion of the obstacle (a sphere) with respect to the background. This changes the wave propagation by a kind of (Doppler–Fizeau) refraction of the wave by the hydrodynamic velocity field around a moving sphere. This velocity field has long been known for a moving sphere: in the low Mach number limit for a perfect fluid it reads

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{u}_0\left(1+\frac{R^3}{2r^3}\right) - \frac{3}{2}(\boldsymbol{u}_0\cdot\boldsymbol{r})\boldsymbol{r}\frac{R^3}{r^5}.$$

This is the velocity field in the reference frame of the sphere of radius R, $\boldsymbol{u}_0$ being the uniform speed of the fluid at infinity. This velocity field is responsible for a 'mirage' effect, that is the bending of the rays (i.e. the lines perpendicular to the isophase surfaces) by the non-uniform flow structure around the obstacle. This mirage is not symmetric with respect to the velocity reversal and induces an imbalance (and a 'quantum drag') between the radiation pressure from phonons coming from opposite directions. The d'Alembert equation for the phonon propagation reads for a medium at rest:

$$\frac{\partial^2\Phi}{\partial t^2} - c_s^2\nabla^2\Phi = 0.$$

But, as noticed in [4], this system is not Lorentz invariant because the original equation (1) is Galilean (as opposed to Lorentz) invariant. For a uniform flow velocity $\boldsymbol{v}$, the d'Alembert equation becomes

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)^2\Phi - c_s^2\nabla^2\Phi = 0. \tag{4}$$

For the mirage problem we are going to consider, we shall deal with a non-uniform velocity field $\boldsymbol{u}(\boldsymbol{r})$. We shall substitute a non-constant $\boldsymbol{u}(\boldsymbol{r})$ for $\boldsymbol{v}$ in (4), an approximation for the propagation of phonons in a non-uniform velocity field, since (4) is strictly true for a uniform velocity field $\boldsymbol{v}$ only. However, in the present problem the drag is dominated by fluctuations of a wavelength much shorter than the radius of the sphere, so (4) may be kept as a first-order approximation in this limit of 'geometrical optics'. At a given frequency ω, the equation of propagation becomes

$$(\mathrm{i}\omega + \boldsymbol{u}\cdot\nabla)^2\Phi - c_s^2\nabla^2\Phi = 0$$

which is solved formally in the WKB/eikonal limit as

$$\Phi = A(\boldsymbol{r})\mathrm{e}^{\mathrm{i}S(\boldsymbol{r})}$$

where the phase $S(\boldsymbol{r})$ is given (at first order in $\beta = u_0/c_s$) as

$$\nabla S(\boldsymbol{r}) = \frac{\omega}{c_s}\left[\boldsymbol{\sigma}' + \frac{\boldsymbol{u}(\boldsymbol{r})}{c_s}\right] \tag{5}$$

where $\boldsymbol{\sigma}'$ is a unit vector (to be defined more precisely later). This gives the deviation of the ray with respect to a straight line. Note also that this perturbative solution is consistent because the velocity field $u(r)$ is irrotational. This allows us to integrate equation (5) at once since $u(r)$ can be written as the gradient of a velocity potential; $S(r)$ is then given formally as

$$S(r) = \frac{\omega}{c_s}\left[\sigma' \cdot r - \frac{\phi}{c_s}\right]$$

where ϕ is the velocity potential ($u(r) = -\nabla\phi, \phi = -u_0 \cdot r(1 + (R^3/2r^3))$ for a sphere). Far from the sphere, the phase field $S(r)$ becomes $k(\sigma' + (u_0/c_s)) \cdot r$ where $k = \omega/c_s$. The Galilean transform does not change the space coordinates (this is the crucial difference between our case and that analysed by Einstein [2] which was concerned with the electromagnetic fluctuations *in vacuo*, being concerned with the Lorentz invariance instead), so then the wavenumber at infinity is $k(\sigma' + (u_0/c_s)$ in either Galilean frame, i.e. the frame of the sphere or the rest frame of the fluid at infinity. Hence, the unperturbed wavenumber must be this wavenumber at infinity, that is $k(\sigma' + (u_0/c_s)$ and one must write

$$\nabla S(r) = \frac{\omega}{c_s}\left[\sigma + \frac{u(r) - u_0}{c_s}\right]$$

to ensure $k = (\omega/c_s)\sigma$ is the unperturbed wavenumber. The gradient of this phase can be taken, again in the eikonal limit, as the local wavenumber needed to compute the tensor component T_{nn} occurring in (3). From (2) we obtain

$$T_{nn} = \frac{2\hbar^2}{m}|\delta\Psi_i|^2[\nabla S(r) \cdot n]^2.$$

We are interested in the contribution to T_{nn} that is linear in u. Note at this step of the calculation that any contribution to $\delta\Psi_i$ (and thus to T_{nn}) linear in u is accounted for by the change of the phase $S(r)$; the contributions arising from the change of amplitude A would be of higher-order either in the gradients of u or in the powers of u. For a plane wave there is also a part of T_{nn} that is independent of the velocity; its contribution to F disappears after integration over all possible orientations of the wavenumber at infinity and by symmetry under space reflection. That part of T_{nn} contributing to F is

$$T_{nn} = \frac{4\hbar^2 k}{m}|\delta\Psi_i|^2[k \cdot n]\left[\frac{u(r) - u_0}{c_s} \cdot n\right]$$

where k is the wavenumber at infinity. The boundary condition for the velocity on the surface of the sphere is $u(r) \cdot n = 0$, whence the result

$$T_{nn} = \frac{4\hbar^2 k}{m}|\delta\Psi_i|^2[k \cdot n]\left[\frac{-u_0}{c_s} \cdot n\right].$$

Now we can perform the integral in (3) to obtain the drag force on a sphere:

$$F = -\frac{8\pi|\delta\Psi_i|^2 k^2 R^2}{3m}\beta \tag{6}$$

where $\beta = u_0/c_s$.

It remains to integrate over all possible values of the wavenumber since we are considering the contribution of phonons arising from the zero-point fluctuations of every mode of the phonon field.

Let Ω be the volume of the system. As the energy of the phonon-mode is a zero-point energy, one has

$$\langle|\delta\Psi_i|^2\rangle = \frac{m\omega_k}{\Omega\hbar k^2} \tag{7}$$

where the average $\langle\cdots\rangle$ is over quantum fluctuations. Substituting this into (6), one obtains the contribution to $\boldsymbol{F}$ from a given phonon-mode. The total 'braking force' is obtained by adding the contributions of all modes. The number of modes dN_k in the interval $[\boldsymbol{k}, \boldsymbol{k}+d\boldsymbol{k}]$ is related to the volume Ω (in the WKB limit) by the Weyl–Poincaré formula:

$$dN_k = \frac{\Omega\, d\boldsymbol{k}}{(2\pi)^3}$$

where $d\boldsymbol{k}$ is the volume element in the wavenumber space. Accordingly, the formal expression for $\boldsymbol{F}$ at first-order in β is

$$\boldsymbol{F} = -\frac{\hbar R^2}{3\pi^2}\beta \int d\boldsymbol{k}\omega_k. \tag{8}$$

Indeed, this integral over $\boldsymbol{k}$ has a very strong divergence (like k^4) at large wavenumbers. However, as explained in [1], this divergence does not actually exist because the assumption of free non-interacting phonons does not hold when the wavelength becomes too short.

It was argued in [1] that the smallest wavelength implied by this phenomenon of 'quantum braking' should be of the order of $(R\lambda_0)^{1/2}$ (λ_0 with microscopic length $= h/mc_s$). This gives as an order of magnitude estimate for $\boldsymbol{F}$

$$\boldsymbol{F} \sim -\frac{h\boldsymbol{v}}{\lambda_0^2}.$$

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